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Structural Analysis and Optimization of Bells Using Finite Elements

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Abstract

This paper deals with the structural analysis and shape optimization of bells. Using modal analysis it is possible to reconstruct the dynamic behavior of the bell by superposition of the excited eigenmodes. An examination of the nodal participation reveals that only a few of these modes are importance to the perceived sound the bell makes. From such consideration, tuning of bells is usually carried out by ensuring that the ratios of the first seven main frequencies (or partials) are in the correct proportions. Frequencies are determined using linear, quadratic and cubic, variable thickness, $C(0)$ continuity, Mindlin-Reissner axisymmetric finite elements. The objective of shape optimization is to ensure that the ratios of the first seven main frequencies of bells are in the correct proportions. Several examples are included to illustrate the application of the analysis and optimization algorithm to the bells.

1. Introduction

In almost every part of the world the sound of bells is familiar to the local population. A bell is essentially a stiff three dimensional shell with a thick wall and is usually axisymmetric. After a bell is excited by its clapper, its vibrating wall excites the air surrounding the bell, thereby radiating the typical bell sound. The sound spectrum of a bell consists of a large number of pure tones, so-called partials, each with a distinct frequency. Every eigenmode vibrates with its own eigenfrequency in a unique mode shape, has an initial amplitude that is determined by the initial conditions of the bell and the excitation process. Every eigenmode (n, m) of a bell is characterized by its typical vibration shape in the vertical and horizontal cross-section. In the horizontal cross-section

$n = 0, 1, 2, 3$, etc. stands for the number of periods (or half-sine waves) in the circumferential direction or half the number of nodal meridians or diameters. Here, $m = 1, 2, 3$, etc. stands for the sequence of modes for each given value of n . The frequencies of notes of D_5 and C_6 church bells are given in a tempered scale. Not all partials in the vibration of bells are equally important. To be important, partials have to be loud and of low order. The frequency ratios of these partials constitute the overtone structure. The highly sophisticated acoustics of these bells will not be discussed any further here, but mention of them serves to illustrate how bells can be analyzed using an axisymmetric shell program.

Following earlier work carried out in Netherlands and Germany [1–4], in this paper we attempt to optimize the design of both minor third and major third church bells by a structural shape optimization algorithm using axisymmetric shell finite elements. Using modal analysis it is possible to construct the dynamic behavior of the bell by superposition of the excited eigenmodes. An examination of the modal participation reveals that only few eigenmodes are of importance to the perceived sound of the bell. From such considerations as these, tuning of church bells is usually carried out by ensuring that the ratios of the first five (or sometimes seven) main frequencies (or partials) are in the correct proportions. In practice, bell founders use a lathe to symmetrically remove material from the inside of the bell at various locations as they tune the partials. For more than half a century, bell founders have tried to find a suitable major third bell profile by a trial and error approach. However, until recently they were not very successful, in spite of considerable experimental effort. The trial and error approach proved to be both time consuming and cumbersome.

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2. Basic optimization algorithm

The structural shape optimization algorithm integrates finite element analysis, cubic spline shape and thickness definition, sensitivity analysis and mathematical programming methods and has been developed for the optimal design of vibrating structures [5,6]. This algorithm was successfully used by Hinton and his co-workers [5,6] for optimal design of axisymmetric and prismatic shell structures. They studied and optimized the oscillatory behaviour of plate and shell structures to avoid resonance. Encouraged by the success of this method for shell structures, we now use this algorithm for the design of bells. The problem of finding optimal bells can be solved using structural shape optimization procedures in which the shape and/or thickness of the bell are varied to achieve specific target frequencies. The structural shape optimization algorithm for bell optimization can be summarized as:

1. Define the initial structural shape of the bell cross-section using splines in terms of a set of design variables, which may include the coordinates, and thicknesses of some specific points, which define the cross-sectional geometry.
2. Generate a mesh of suitable finite elements.
3. Carry out a finite element analysis using curved, variable thickness, $C(0)$ Mindlin-Reissner axisymmetric shell elements [5] and evaluate the first seven important frequencies of the bell.
4. Evaluate the sensitivities of the frequencies with respect to design variables.
5. Using a suitable optimization algorithm, check the current eigenvalues of the bell. If the current bell shape is not optimal, generate a new structural shape and go to step 2, otherwise stop.

3. Optimization problem definition

A shape optimization problem may be summarized in the formal mathematical language of nonlinear programming as follows: Find the design vector $\mathbf{s}$ which *maximizes* (or *minimizes*) the objective function $F(\mathbf{s})$ *subject to* the behavioral constraints $g_j(\mathbf{s}) \leq 0$, equality constraints $h_k(\mathbf{s}) = 0$ and explicit geometric constraints $s_i^l \leq s_i \leq s_i^u$. The subscripts j , k and i denote the number of behavioral constraints, equality constraints and design variables respectively. The terms s_i^l and s_i^u refer to the specified lower and upper bounds on the design variables.

The major design aspect for a bell is to preserve high tonal quality. The objective is therefore to ensure that the ratios of the first seven main frequencies are in the correct proportions. Consequently, the objective function for the bell can be defined as

$$F(\mathbf{s}) = \sum_{i=1}^7 \frac{(\omega_i - \omega_{io})^2}{\omega_{io}^2} a_i \quad (1)$$

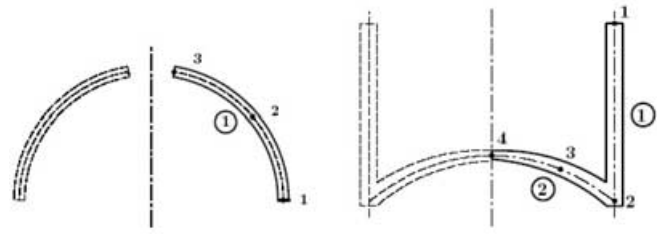


Fig. 1. Representation of geometry of the shells of revolution.

where $F(\mathbf{s})$ is the objective function to be minimized, ω_i is the frequency, ω_{io} is the target frequency and a_i is the weighting factor. When all the frequency requirements are met exactly at the optimal solution, calculated frequencies become equal to the target frequencies and the objective function is equal to zero. Note that, a suitable choice of the weighting factors is important for the convergence of the solution.

In structural optimization, various mathematical programming techniques have been extensively applied. In the present work we use Sequential Quadratic Programming, which is considered to be quite powerful for a wide class problems [7].

4. Geometric representation

As we wish to deal with bells with arbitrary and complex cross-sectional shapes, it necessary to employ a computer aided design tool, which can represent the geometry in a convenient way and allow a subsequent discretization of the structural cross-section prior to the finite element analysis. To do this, we first represent the cross-section of the structure using parametric cubic splines, the cross-sections of typical shells of revolution are shown in Figure 1 and they are representing by either a single cubic spline segment or an assembly of such segments. Each segment is a cubic spline curve passing through certain key points all of which lie on the midsurface of the structure cross-section. Some key points are common to different segments at their points of intersection. To represent a straight line the analyst has to provide a minimum of two key points lying on the segment; for a curved line three points are needed. An automatic mesh generator is then used to generate a mesh of suitable finite elements with mesh densities specified at key points.

In structural shape optimization, finite element meshes must be generated automatically for not only the initial structural shape but also for all the new shape produced during the optimization process.

Selection of shape design variables: To define the structural shape, recent research has emphasized the selection of so called natural design variables [8], which leads to a considerable decrease in the number of design variables and has

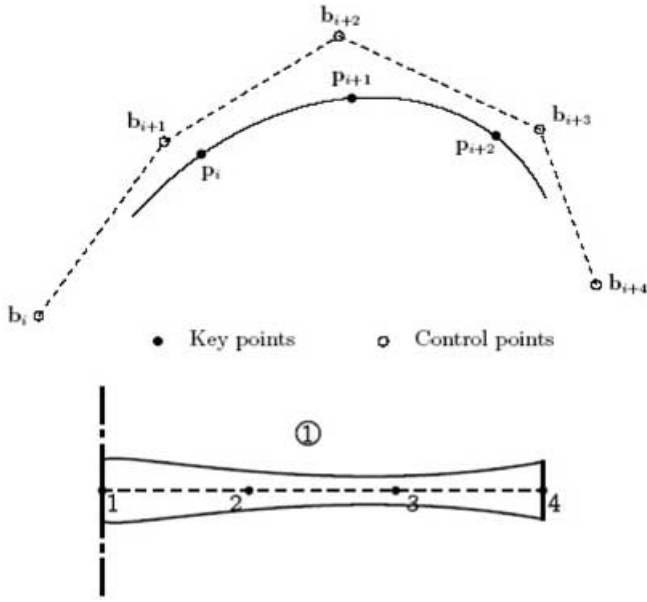


Fig. 2. Selection of the design variables: position vectors of the key points as shape design variables (top) and thickness of the key points as thickness design variables (bottom).

been found to give more realistic shapes. In the present approach the position vectors of the key points used to define the shell midsurfaces are taken as the design variables and can be selected at the discretion of the analyst. Therefore the design variables are no longer the position vectors of the control points as in reference [8] but the position vectors of the key points, which are used to define the shell midsurface.

Figure 2 shows a general curve which passes through the points P_i which have position vectors $\mathbf{p}_i$ where

$$\mathbf{p}_i = [p_{xi}, p_{yi}]^T \quad (2)$$

The position vector of the vertices or control points of the polygon defining this curve are designated $\mathbf{b}_i$. However, in the present approach the position vectors $\mathbf{p}_i$ are taken as design variables instead of the position vectors of the control vertices $\mathbf{b}_i$. We can now define the shape variables $\mathbf{s}$ as

$$\mathbf{s} = [p_{x1}, p_{y1}, p_{x2}, p_{y2}, \dots, p_{xn}, p_{yn}]^T \quad (3)$$

where the position vectors are usually defined with respect to the Cartesian coordinate system as indicated in expression (3) – however, locally defined polar coordinates are sometimes used for convenience.

Selection of thickness design variables: In previous research work on shells only uniform or piecewise uniform thickness variation has been allowed [8]. This overconstrains the optimisation process and does not give the greatest opportunity for weight reduction. In this work, the use of cubic splines for thickness distribution along the shell gives

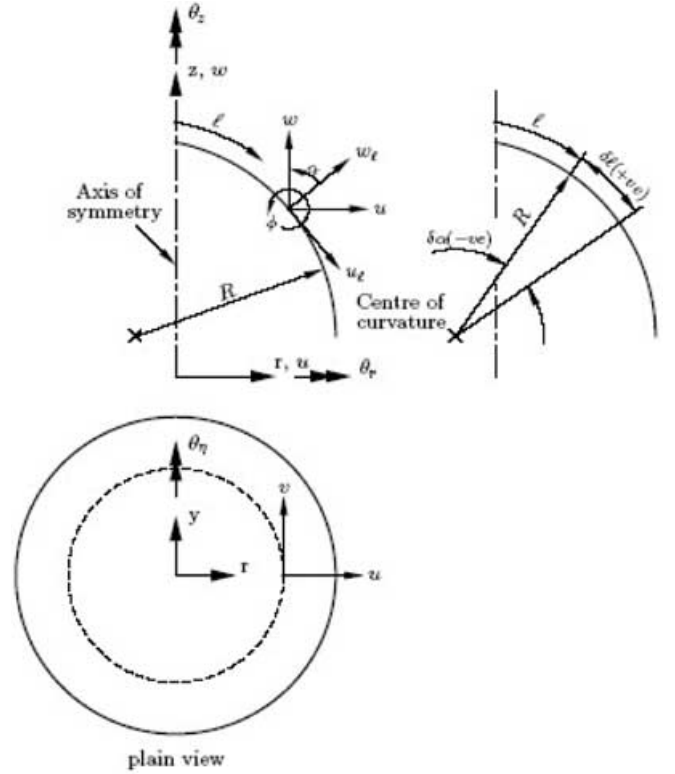


Fig. 3. Definition of terms and axes for axisymmetric Mindlin Reissner shell.

greater flexibility and a smooth variation of thickness. A similar approach to that adopted for the shape design variables is used in which the thickness values at some key points are specified as design variables.

6. Structural analysis and finite element formulation

Fundamental theory: Consider the free vibrations of the Mindlin Reissner curved axis-symmetric shell element shown in Figure 3. Displacement components u_i , v_i and w_i are translations in the local l , η and n directions respectively. Note that η varies from an angle 0 to 2π along a curve of radius r . The displacement components u_i and w_i may be written in terms of global displacements u and w in the r and z directions as

$$u_\ell = u \cos \alpha + w \sin \alpha \quad \text{and} \quad w_\ell = -u \sin \alpha + w \cos \alpha \quad (4)$$

where α is the angle between the r and l axes; see Figure 3. The radius of curvature R may be obtained from the expression

$$\frac{d\alpha}{d\ell} = -\frac{1}{R}. \quad (5)$$

Note that the displacement components v_g and v coincide.

In the absence of external loads and damping effects, the virtual work (or more precisely the virtual power) for a typical curved axisymmetric Mindlin Reissner shell element e is given in terms of the global displacements u , v and w and the rotations ϕ and φ of the midsurface normal in the In and ηn planes respectively by the expression

$$I_e = \int_0^{2\pi} \int_{r_c} (\delta \mathbf{e}_m^T \mathbf{D}_m \mathbf{e}_m + \delta \mathbf{e}_b^T \mathbf{D}_b \mathbf{e}_b + \delta \mathbf{e}_s^T \mathbf{D}_s \mathbf{e}_s) r d\ell d\eta \quad (6)$$

where the membrane strains $\mathbf{e}_m$ are given by

$$\mathbf{e}_m = \left[\frac{\partial u}{\partial \ell} \cos \alpha + \frac{\partial w}{\partial \ell} \sin \alpha, \left(u + \frac{\partial v}{\partial \eta} \right) \frac{1}{r}, \left(\frac{\partial u}{\partial \eta} \cos \alpha + \frac{\partial w}{\partial \eta} \sin \alpha - v \cos \alpha \right) \frac{1}{r} + \frac{\partial v}{\partial \ell} \right]^T \quad (7)$$

The bending strains or curvatures $\mathbf{e}_b$ are given by

$$\mathbf{e}_b = \left[\frac{\partial \phi}{\partial \ell}, - \left(\frac{\partial \phi}{\partial \eta} + \phi \cos \alpha \right) \frac{1}{r}, \left(- \frac{\partial \phi}{\partial \eta} + \phi \cos \alpha + \frac{\partial v}{\partial \ell} \sin \alpha \right) \times \frac{1}{r} - \frac{\partial \phi}{\partial \ell} - \frac{d\alpha}{d\ell} \left(\frac{\partial u}{\partial y} \cos \alpha + \frac{\partial w}{\partial y} \sin \alpha \right) \frac{1}{r} \right] \quad (8)$$

The transverse shear strains $\mathbf{e}_s$ may be written as

$$\mathbf{e}_s = \left[\frac{\partial u}{\partial \ell} \sin \alpha + \frac{\partial w}{\partial \ell} \cos \alpha - \phi, \left(\frac{\partial u}{\partial \eta} \sin \alpha + \frac{\partial w}{\partial \eta} \cos \alpha + v \cos \alpha \right) \frac{1}{r} - \varphi \right]^T \quad (9)$$

For an isotropic material of elastic modulus E , Poisson's ratio ν and thickness t , the matrix of membrane, bending and shear rigidities has the form

$$\mathbf{D}_m = \frac{Et}{(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu/2) \end{bmatrix}$$

$$\mathbf{D}_b = \frac{Et^3}{12(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu/2) \end{bmatrix}$$

and $\mathbf{D}_s = \frac{\kappa^2 Et}{2(1+\nu^2)} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (10)$

where κ is the shear modification factor and is usually taken as 5/6 for an isotropic material.

$$\mathbf{u} = [u, v, w, \phi, \varphi]^T \quad \mathbf{P} = \rho t \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & t^2/12 & 0 & 0 \\ 0 & 0 & 0 & t^2/12 & 0 \\ 0 & 0 & 0 & 0 & t^2/12 \end{bmatrix} \quad (11)$$

where ρ is the density of the material – thus rotatory inertia effects are included. The vector $\mathbf{\ddot{u}}$ contains the accelerations of the displacement components u , v and w as well as the

normal rotations ϕ and φ – a superposed dot implies differentiation with respect to time

$$u(\ell, \eta) = \sum_{p=1}^h u^p(\ell) C_p \quad v(\ell, \eta) = \sum_{p=1}^h v^p(\ell) S_p$$

$$w(\ell, \eta) = \sum_{p=1}^h w^p(\ell) C_p \quad \phi(\ell, \eta) = \sum_{p=1}^h \phi^p(\ell) C_p$$

$$\psi(\ell, \eta) = \sum_{p=1}^h \psi^p(\ell) S_p \quad (12)$$

where $C_p = \cos(p\eta)$ and $S_p = \sin(p\eta)$, u^p , v^p , w^p , ϕ^p and ψ^p are displacement and rotation amplitudes for the p^{th} harmonic term and h is the number of harmonic terms used in the analysis.

The next step is to discretise the displacement and rotation amplitudes (which are functions of the ℓ – coordinate only) using an n -noded finite element representation so that within element e the amplitudes can be written as

$$u^p(\ell) = \sum_{i=1}^n N_i u_i^p; \quad v^p(\ell) = \sum_{i=1}^n N_i v_i^p; \quad w^p(\ell) = \sum_{i=1}^n N_i w_i^p$$

$$\phi^p(\ell) = \sum_{i=1}^n N_i \phi_i^p; \quad \psi^p(\ell) = \sum_{i=1}^n N_i \psi_i^p; \quad (13)$$

where u^p , v^p , w^p , ϕ^p and ψ^p are typical nodal degrees of freedom associated with node i and harmonic p . For convenience, these terms are grouped together so that

$$\mathbf{d}_i^p = [u_i^p, v_i^p, w_i^p, \phi_i^p, \psi_i^p]^T \quad (14)$$

$N_i(\xi)$ is the shape function associated with node i . These elements are essentially isoparametric so that

$$r = \sum_{i=1}^n N_i r_i; \quad z = \sum_{i=1}^n N_i z_i; \quad t = \sum_{i=1}^n N_i t_i \quad (15)$$

where r_i and z_i are typical coordinates of node i and t_i is the thickness at node i .

If we list the nodal displacements and accelerations in vectors $\mathbf{d}$ and $\mathbf{\ddot{d}}$ respectively, then upon substitution of (10)–(15) into (6) for all the elements we obtain the expression

$$\delta \mathbf{d} = [\mathbf{K} \mathbf{d} + \mathbf{M} \mathbf{\ddot{d}}] = 0 \quad (16)$$

where $\mathbf{K}$ and $\mathbf{M}$ are the global stiffness and mass matrices respectively and contain submatrices contributed from each element e linking nodes i and j and harmonics p and q . These submatrices have the form

$$[\mathbf{K}_{ij}^e]^{pq} = \int_0^{2\pi} \int_{-1}^{+1} \left\{ [\mathbf{B}_{mi}^p]^T \mathbf{D}_m \mathbf{B}_{mj}^q + [\mathbf{B}_{bi}^p]^T \mathbf{D}_b \mathbf{B}_{bj}^q + [\mathbf{B}_{si}^p]^T \mathbf{D}_s \mathbf{B}_{sj}^q \right\} r J d\xi d\eta \quad (17)$$

$$[\mathbf{M}_{ij}^e]^{pq} = \int_0^{2\pi} \int_{-1}^{+1} \left\{ [\mathbf{N}_i^p]^T \mathbf{P} \mathbf{N}_j^q \right\} r J d\xi d\eta \quad (18)$$

where typically

$$\mathbf{B}_{mi}^p = \begin{bmatrix} (dN_i/d\ell)C_p \cos\alpha & 0 & (dN_i/d\ell)C_p \sin\alpha & 0 & 0 \\ (N_i/r)C_p & -(pN_i/r)C_p & 0 & 0 & 0 \\ (pN_i/r)S_p \cos\alpha & (dN_i/d\ell - (N_i/r)\cos\alpha)S_p & (pN_i/r)S_p \sin\alpha & 0 & 0 \end{bmatrix}$$

$$\mathbf{B}_{bi}^p = \begin{bmatrix} 0 & 0 & 0 & -(dN_i/d\ell)C_p & 0 \\ 0 & 0 & 0 & -(N_i/r)C_p \cos\alpha & (pN_i/r)C_p \\ (p(N_i/r) & dN_i/d\ell(1/r) & (p(N_i/r) & -(pN_i/r)S_p & ((N_i/r)\cos\alpha \\ \times S_p \cos\alpha)/R & \times S_p \sin\alpha & \times S_p \sin\alpha)/R & & -dN_i/d\ell)S_p \end{bmatrix}$$

$$\mathbf{B}_{si}^p = \begin{bmatrix} -(dN_i/d\ell)C_p \sin\alpha & 0 & (dN_i/d\ell)C_p \cos\alpha & -N_i C_p & 0 \\ -(pN_i/r)S_p \sin\alpha & (N_i/r)S_p \sin\alpha & (pN_i/r)S_p \cos\alpha & 0 & -N_i S_p \end{bmatrix}$$

and

$$\mathbf{N}_i^p = N_i \begin{bmatrix} S_p & 0 & 0 & 0 & 0 \\ 0 & C_p & 0 & 0 & 0 \\ 0 & 0 & S_p & 0 & 0 \\ 0 & 0 & 0 & S_p & 0 \\ 0 & 0 & 0 & 0 & C_p \end{bmatrix} \quad (19)$$

$[\mathbf{K}_{ij}^e]^{pg}$ and $[\mathbf{M}_{ij}^e]^{pg} = 0$ if $p \neq q$ because of the orthogonality conditions. Since the discretised virtual power expression (16) must be true for any set of virtual displacements $\delta \mathbf{d}^p$, then by taking advantage of the orthogonality conditions, it may be re-written in uncoupled form for each harmonic p as

$$\mathbf{K}^{pp} \mathbf{d}^p - \mathbf{M}^{pp} \ddot{\mathbf{d}}^p = \mathbf{0} \quad (20)$$

Note that when $p = 0$ we are solving for axisymmetric vibrations. Matrix $[\mathbf{M}_{ij}^e]^{pp}$ is independent of the harmonic number p and therefore, the same matrix can be used for all the different harmonic equations.

7. Sensitivity analysis

The mathematical programming method requires the derivatives of the objective function $F(\mathbf{s})$ and the derivative of the behavioral constraints $g(\mathbf{s})$ with respect to design variables $\mathbf{s}$. Haftka and Adelman [9] have given an excellent survey paper on the sensitivity analysis of static, transient, and eigenvalue problems and address the associated computational difficulties and computational errors. The frequency derivatives may be approximated in an efficient way using a semi-analytical method.

Derivatives of natural frequencies: The equations governing the free vibrations of shells using finite element method is given in (16) where for the p^{th} harmonic $[\mathbf{K}]^{pp}$ is the stiffness matrix, $[\mathbf{M}]^{pp}$ is the mass matrix, ω_p is the frequency and $\bar{\mathbf{d}}^p$ is the vibration mode shape which is normalized so that

$$\bar{\mathbf{d}}_p^T \mathbf{M} \bar{\mathbf{d}}_p = 1 \quad (21)$$

When the eigenvalues are distinct, the expression for the frequency derivative with respect to the design variable s_k can be derived from (16) and (21) so that

$$\frac{\partial \omega}{\partial s_k} = \frac{1}{2\omega_p} \mathbf{d}_p^T \left(\frac{\partial \mathbf{K}^{pp}}{\partial s_k} - \omega_p^2 \frac{\partial \mathbf{M}^{pp}}{\partial s_k} \right) \mathbf{d}_p \quad (22)$$

Derivatives of the stiffness and mass matrices are obtained by a forward finite difference. These derivatives are computed by re-calculating $\mathbf{K}^{pp}$ and $\mathbf{M}^{pp}$ for a small perturbation Δs_k of the design variable (coordinates and/or thicknesses). The derivatives of the stiffness and mass matrices with respect to the design variable s_k may then be written as

$$\frac{\partial \mathbf{K}^{pp}}{\partial s_k} \approx \frac{\mathbf{K}^{pp}(s_k + \Delta s_k) - \mathbf{K}^{pp}(s_k)}{\Delta s_k}$$

and

$$\frac{\partial \mathbf{M}^{pp}}{\partial s_k} \approx \frac{\mathbf{M}^{pp}(s_k + \Delta s_k) - \mathbf{M}^{pp}(s_k)}{\Delta s_k} \quad (23)$$

7. Examples bell analysis and optimization

7.1 Codnor hand bell

This example is taken from the work of Charnley [10]. The geometry of the Codnor hand bell is shown in Figure 4. The bell is completely fixed at the top and the material properties adopted are: elastic modulus $E = 103 \text{ GPa}$, density $\rho = 8850 \text{ kg/m}^3$ and Poisson's ratio $\nu = 0.38$.

The experimental analysis is carried by Charnley [10]. The finite element analysis of the bell is carried out using 32 cubic elements. The results and corresponding experimental

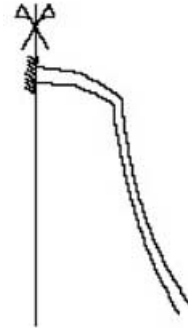


Fig. 4. Geometry of Codnor hand bell.

values are shown in Table 1. The difference between experimental analysis and present solution is less than 5% except for the fifth eigenvalues in which case the difference is around 12%. The modal amplitudes of the different partials are shown in Figure 5.

7.2 D₅ bell

The bell used in this analysis is the same as that used by Perrin et al. [11] in their experiments and its geometry is shown in Figure 6. The bell which Perrin used in his experiment had diameter 70.2cm, height 56.6cm and weight

Table 1. Frequencies (in Hertz) of Codnor hand bell.

<i>m,n</i>	MR element cubic	experimental [10]
2,1	262.2	262.3
3,1	767.0	726.0
3,2	1397.2	1245.0
4,1	1350.7	1397.0
5,1	1867.2	1829.0

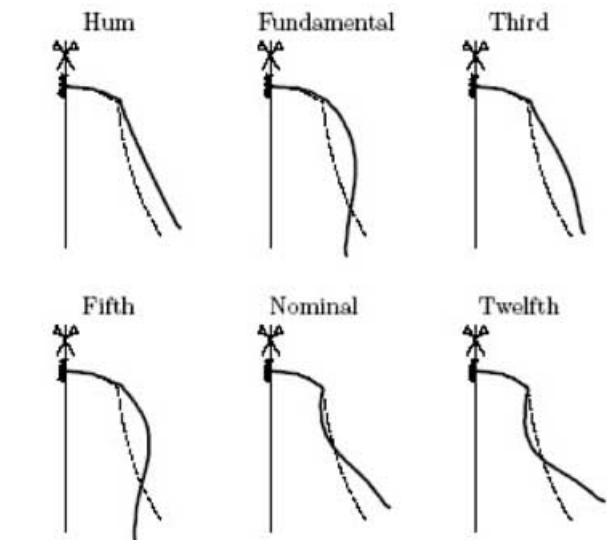


Fig. 5. Modal amplitudes of Codnor hand bell.

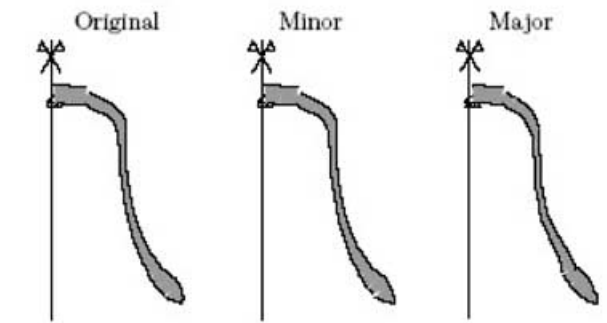


Fig. 6. Original shape and optimal minor-third and major-third D₅ church bells.

214kg. The material properties of the bell are elastic modulus $E = 103\text{ GPa}$, density $\rho = 8850\text{ kg/m}^3$ and Poisson's ratio $\nu = 0.38$. It was cast by Messrs. John Taylor of Loughborough, being destined for St. Margaret's Church, Leigh-on-Sea, Essex, and its musical note is D₅. The bell is completely fixed against vertical movement at its top.

a-) **Analysis:** The analysis is carried out using 37 cubic elements with a total of 559 degrees of freedom. The results of the analysis in the form of frequency ratios, together with the corresponding experimental values [11] are shown in Table 2. Only the frequencies, which are important in the sound spectrum of the bell, are listed and corresponding modal amplitudes are given in Figure 7. Table 2 illustrates that there is some slight discrepancy between the reference solutions [11] and the present results. However, generally good agreement is observed. The maximum percentage difference between the experimental and present solution is less than 2.3%.

b-) **Shape optimization:** The target frequencies for the minor third and major third bells are given in Table 2. Only shape design variables are considered and coordinates of the midsurface are defined as design variables. Two different cases are considered as follows:

- *minor third bell optimization:* 13 key points on the midsurface of the bell are defined as design variables which are allowed to move in radial direction.
- *major third bell optimization:* Coordinates of the 11 key points are defined as design variables. Key points are allowed to move both radial and horizontal directions, so that the total number of design variables used is 22.

Target, initial and optimal frequencies of the minor third and major third bells are given in Table 2. Optimum solutions for the minor third and major third bells are shown in

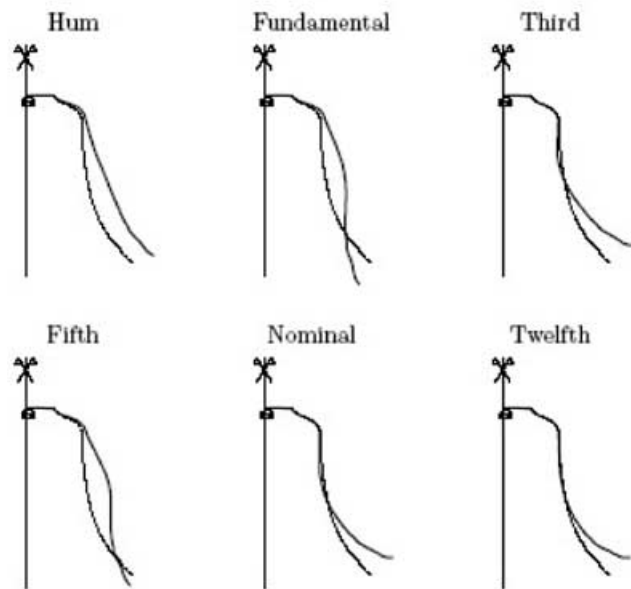


Fig. 7. Modal amplitudes of Taylor D₅ major-third church bell.

Table 2. Frequencies in D₅ minor third and major third church bells.

partial				initial analysis (minor)		optimal results	
name	code	note	target	experiment	axishell FE	minor	major
hum	2,1	D ₄	293.66 (1.000) [†]	292.72 (1.000)	294.88 (1.000)	293.97 (1.000)	293.27 (1.000)
fundamental	2,2	D ₅	587.33 (2.000)	585.92 (2.001)	594.68 (2.019)	593.73 (2.020)	591.87 (2.018)
third	3,1	F ₅ /F ₅ [#]	698.46/739.99 [‡] (2.378/2.531)	692.94 (2.367)	693.23 (2.353)	695.16 (2.365)	738.03 (2.516)
fifth	3,2	A ₅	880.0 (2.997)	882.53 (3.015)	891.93 (3.028)	892.52 (3.036)	892.67 (3.044)
nominal (octave)	4,1	D ₆	1174.7 (4.000)	1172.0 (4.004)	1169.92 (3.971)	1167.46 (3.971)	1180.90 (4.027)
twelfth	5,1	A ₆	1760.0 (5.993)	1764.3 (6.027)	1750.87 (5.944)	1754.61 (5.969)	1768.29 (6.030)
double octave	6,1	D ₇	2349.3 (8.000)	2441.2 (8.340)	2403.64 (8.159)	2386.81 (8.119)	2397.74 (8.176)

[†] Values in brackets are frequency ratios normalized with respect to hum values.

[‡] First value is for a minor third bell, second value is for a major third bell.

Figure 6 and obtained after 23 and 28 iterations respectively. The maximum error in the frequencies is 2.3% and occurs at the double octave frequency before optimization. The error in the nominal frequency reduces to 1.6% in the optimum solution for the minor third bell.

7.3 Minor third C₆ bell

This example is taken from the work of Bletzinger and Reitingner [3]. The geometry of the minor third C₆ bell is shown in Figure 8.

a-) Analysis: The bell is analyzed using 25 cubic elements with a total of 384 degrees of freedom. The results are compared with those obtained by Bletzinger and Reitingner and are given in Table 3. The modal amplitudes of the important mode shapes are given in Figure 9. Generally, good agreement with the reference solution [3] is observed with very good frequency ratios obtained.

b-) Shape optimization: Based on the geometry given in Figure 8 we now perform the shape optimization for an improved minor third bell and also a major third bell. Target frequencies for these bells are given in Table 3. Only shape design variables are considered in this example. Design variables are allowed to move within 15% of their initial values. Two different cases are considered:

- *minor third bell optimization:* The radial coordinates of the 9 key points are used as design variables.
- *major third bell optimization:* 18 design variables are used to define the geometry. These design variables are the coordinates of the first seven key points on the mid-surface of the bell.

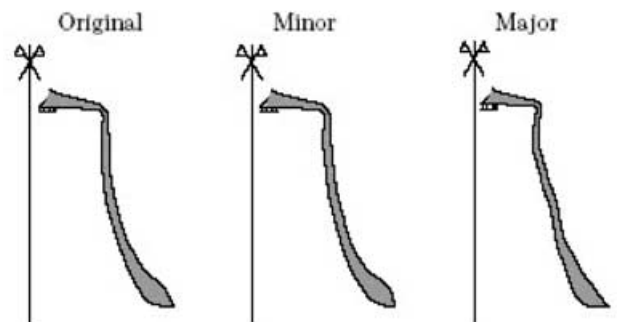


Fig. 8. Original shape and optimal minor-third and major-third C₆ church bells.

Table 3 presents the target frequencies and initial and optimum minor-major third bell frequencies. The optimum shapes are obtained in 16 and 29 iterations for the minor and major third bells respectively and are shown in Figure 8. The maximum errors between the optimum and target frequencies are 1.0% and 1.6% for the minor third and major third bells.

4. Conclusions

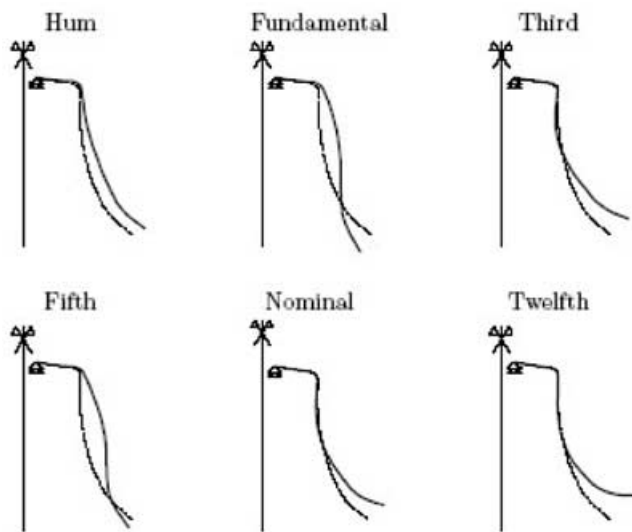
In this paper an efficient and flexible, automated approach to analysis and shape optimization of bells has been presented. Several examples have been studied and used to demonstrate the capabilities offered by this computational tool. Some conclusion are now drawn from this study:

Table 3. Frequencies in C₆ minor third and major third church bells.

partial				initial analysis (minor)		optimal results	
name	code	note	target	experiment	axisshell FE	minor	major
hum	2,1	C ₄	523.25 (1.000) [†]	512.9 (1.000)	524.12 (1.000)	523.74 (1.000)	524.56 (1.000)
fundamental	2,2	C ₆	1046.5 (2.000)	1004.0 (1.957)	1058.5 (2.019)	1052.2 (2.009)	1056.6 (2.014)
third	3,1	E ₆ ^b /E ₆	1244.5/1318.6 [‡] (2.378/2.520)	1239.0 (2.416)	1239.6 (2.365)	1247.5 (2.382)	1327.7 (2.531)
fifth	3,2	G ₆	1568.0 (2.997)	1534.2 (2.991)	1600.7 (3.054)	1584.2 (3.025)	1593.9 (3.039)
nominal (octave)	4,1	C ₇	2093.0 (4.000)	2064.8 (4.026)	2062.6 (3.935)	2075.1 (3.962)	2083.7 (3.972)
twelfth	5,1	G ₇	3136.0 (5.993)	—	3112.1 (5.938)	3125.4 (5.967)	3119.8 (5.947)
double octave	6,1	C ₈	4186.3 (8.000)	—	4155.3 (7.928)	4161.0 (7.945)	4207.3 (8.021)

[†] Values in brackets are frequency ratios normalized with respect to hum values.

[‡] First value is for a minor third bell, second value is for a major third bell.

Fig. 9. Modal amplitudes of Taylor C₆ major-third church bell.

- The optimization problem should be posed correctly otherwise unrealistic solutions may be found. Consequently, care should be taken in the selection of appropriate design variables and the accurate representation of the geometry.
- The results obtained using finite element analysis tool generally compare well with those obtained from other sources such as experimental method, axisymmetric three dimensional finite element analysis and shell elements.
- The results illustrate that the shape optimization algorithm presented here can be used with confidence for the design and tuning of the bells.

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